Math 10A HW10 Solutions

(1) True, changing initial conditions affects the constants

(2) False, can simply substitute $a_n = f(n)$ into recurrence relation to see if it's a solution

(3) $a_n = a_{n-1} + 2a_{n-2}$ $\quad a_0 = 2, a_1 = 1$

$\lambda^2 = \lambda + 2$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

$a_n = c_1 2^n + c_2(-1)^n$

Using initial conditions we see:

$c_1 + c_2 = 2 \Rightarrow c_2 = 2 - c_1 \Rightarrow c_2 = 2 - 1 = 1$

$2c_1 - c_2 = 1$

$\Rightarrow 2c_1 - (2 - c_1) = 1$

$2c_1 - 2 + c_1 = 1$

$3c_1 = 3$

$c_1 = 1$

$a_n = 2^n + (-1)^n$

(4) $a_n = 2a_{n-1} + 3a_{n-2}$

$\lambda^2 = 2\lambda + 3$
$\lambda^2 - 2\lambda - 3 = 0$
$(\lambda - 3)(\lambda + 1) = 0$
$\lambda = 3, -1$
$a_n = c_1(3^n) + c_2(-1^n)$

(5) $a_n = -6a_{n-1} - 9a_{n-2}$ $\quad a_0 = 3, a_1 = -3$

$\lambda^2 = -6\lambda - 9$
$\lambda^2 + 6\lambda + 9 = 0$
$(\lambda + 3)^2 = 0$
$\lambda = -3$
$a_n = c_1(-3^n) + c_2 n(-3^n)$

Using initial conditions we have:
$c_1 = 3$
$c_1(-3) + c_2(-3) = -3$
$\Rightarrow -9 - 3c_2 = -3$
$-3c_2 = 6$
$c_2 = -2$

$a_n = 3(-3^n) - 2n(-3^n)$

(6) $a_n = 4a_{n-2}$ $\quad a_0 = 0,\ a_1 = 4$

$\lambda^2 = 4$

$\lambda = \pm 2$

$a_n = c_1 2^n + c_2(-2)^n$

Using initial conditions we have :

$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 = 1$

$2c_1 - 2c_2 = 4$

$\Rightarrow -2c_2 - 2c_2 = 4$

$\Rightarrow -4c_2 = 4$

$c_2 = -1$

$a_n = 2^n - (-2)^n$

(7) Need $(\lambda-3)(\lambda-2) = 0$

$\Rightarrow \lambda^2 - 5\lambda + 6$

$\Rightarrow a_n - 5a_{n-1} + 6a_{n-2} = 0$

(8) $a_n = n-2 = n\cdot 1^n - 2\cdot 1^n$

$\Rightarrow$ double root of $\lambda = 1$

$\Rightarrow (\lambda-1)^2 = \lambda^2 - 2\lambda + 1$

$\Rightarrow a_n - 2a_{n-1} + a_{n-2} = 0$

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(9) True!

$$y = \frac{\ln(x)}{x} \Rightarrow y' = \frac{x \cdot \frac{1}{x} - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$$

So we see $x^2 y' + xy = x^2\left(\frac{1-\ln x}{x^2}\right) + x\left(\frac{\ln(x)}{x}\right)$

$= 1 - \ln(x) + \ln(x)$

$= 1$

(10) $y = te^t + 1$

$y' = te^t + e^t$

$y'' = te^t + e^t + e^t = (t+2)e^t$

So we see $y'' - 2y' = (t+2)e^t - 2(te^t + e^t)$

$= te^t + 2e^t - 2te^t - 2e^t$

$= -te^t$

$= 1 - y$

(11) $y = 2e^{(2t)^{-1}}$

$y' = 2e^{(2t)^{-1}} \cdot (-1)(2t)^{-2} \cdot 2 = -4e^{(2t)^{-1}} / (2t)^2$

$\Rightarrow y' = -e^{(2t)^{-1}} / t^2$

$2t^2 y' + y = 2t^2\left(\frac{-e^{1/2t}}{t^2}\right) + 2e^{1/2t}$

$= -2e^{1/2t} + 2e^{1/2t}$

$= 0$

(12)

(a) $y'' = 2y$
2nd order, homogeneous, linear,
constant coefficients

(b) $y' = y^2$
1st order, homogeneous, not linear,
constant coefficients

(c) $y' + (e^t - \sin t)y = \tan t$
1st order, nonhomogeneous, linear,
does not have constant coefficients

(d) $y' - ty = t^2$
1st order, nonhomogeneous, linear,
does not have constant coefficients

(13) True, refer to worksheet

(14) True, refer to worksheet

(15) True, refer to worksheet

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(16)

(a) $y'' + 2y' - 3y = 0$

$\Rightarrow$ use characteristic polynomial so we have:

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$r = 1, -3$

$\Rightarrow$ general solution is $y(t) = c_1 e^{t} + c_2 e^{-3t}$

(b) $y'' + 5y' = 0$

$\Rightarrow$ use characteristic polynomial so we have:

$r^2 + 5r = 0$

$r(r+5) = 0$

$r = 0, -5$

$\Rightarrow$ general solution is $y(t) = c_1 e^{-5t} + c_2$

(c) $y'' + 2y' + 5y = 0$

$\Rightarrow$ use characteristic polynomial so we have:

$r^2 + 2r + 5 = 0$

$r = \dfrac{-2 \pm \sqrt{4 - 4(5)}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2}$

$\Rightarrow$ general solution is $y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

(17) $y'' - 3y' + 2y = 0$; $y(0) = 0$, $y'(0) = 2$

Using characteristic polynomial,

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 1, 2$

Solution is of the form $y(t) = c_1 e^t + c_2 e^{2t}$.

Using initial conditions we see:

$y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y'(t) = c_1 e^t + 2c_2 e^{2t}$

$\Rightarrow y'(0) = c_1 + 2c_2 = 2$

$\Rightarrow -c_2 + 2c_2 = 2$

$c_2 = 2$

$c_1 = -2$

Thus solution is $y(t) = -2e^t + 2e^{2t}$

~~(18)~~ $\cancel{y'' + 4y' = 0}$; $\cancel{y(0) = 0}$, $\cancel{y(\pi) = 1}$

~~Using characteristic polynomial~~,

$\cancel{r^2 + 4r = 0}$

$\cancel{r(r+4) = 0}$

$\cancel{r = 0, -4}$

~~Solution is of the form~~ $\cancel{y(t) = c_1 e^{-4t}} + c_2$.

~~Using boundary conditions we see~~:

$\cancel{y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2}$

$\cancel{y(\pi) =}$

(18) $y'' + 4y = 0 \qquad y(0) = 0,\ y(\pi) = 1$

Using characteristic polynomial,

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

Solution is of the form $y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

Using boundary conditions we see:

$y(0) = c_1 = 0 \quad \rightarrow \leftarrow$

$y(\pi) = c_1 = 1$

No solutions!

(19) We want $r = 2, -3$ so characteristic polynomial is $(r-2)(r+3) = 0$

$\Leftrightarrow r^2 + 3r - 2r - 6 = 0$

$\Leftrightarrow r^2 + r - 6 = 0$

Thus the 2nd order linear DE must be

$y'' + y' - 6y = 0$.

(20) $y = e^{3t} \sin(t) \Rightarrow$ roots $3 \pm i$

$\Rightarrow (r-(3+i))(r-(3-i)) = (r-3-i)(r-3+i)$

$\Rightarrow r^2 - 6r + 10$

$\Rightarrow y'' - 6y' + 10y = 0$

where $y(0) = 0$

$y'(0) = 1$

(21) True!

(22) True by definition of linear &
separable!

(23) True, need to find remaining
solutions by solving $f(y)=0$.

(24)
(a) $y' = t + 5y$

Rewriting this we see $y' - 5y = t$ so
we'll use integrating factors to solve.

$\Rightarrow I(t) = e^{\int -5\,dt} = e^{-5t}$

$\Rightarrow y = \frac{1}{e^{-5t}} \int e^{-5t} t\,dt$ (integration by parts)

$y = \frac{1}{\cancel{e^{-5t}}} \left(\frac{-1}{25} \cancel{e^{-5t}} (5t+1) + C \right)$

$y = \frac{-1}{25}(5t+1) + \frac{C}{e^{-5t}}$

(b) $y' + y = \sin(e^x)$

Using integrating factors we see that
$I(x) = e^{\int dx} = e^x$

$\Rightarrow y = \frac{1}{e^x} \int e^x \sin(e^x)\,dx$ (U-sub)

$y = \frac{1}{e^x}(-\cos(e^x) + C)$

(c) $y' - 2ty = 3t^2 e^{t^2}$, $y(0) = 5$

Using integrating factors we have
$I(t) = e^{\int -2t\,dt} = e^{-t^2}$

$\Rightarrow y = \frac{1}{e^{-t^2}} \int \cancel{e^{-t^2}} 3t^2 \cancel{e^{t^2}}\,dt = \frac{1}{e^{-t^2}} \int 3t^2\,dt$

$y = \frac{3}{e^{-t^2}}\left(\frac{t^3}{3} + C\right) = \frac{t^3}{e^{-t^2}} + Ce^{t^2}$

$y(0) = C = 5 \Rightarrow y(t) = t^3 e^{t^2} + 5e^{t^2}$

(d) $y' = \frac{1}{x} y + x\sin(x) \qquad y(\pi) = 0$

Using integrating factors we see that
$I(x) = e^{\int \frac{1}{x} dx} = e^{-\ln(x)} = 1/x$

$y = x \int \sin(x)\,dx = x\,(-\cos x + C)$

$y(\pi) = -\pi\cos(\pi) + \pi C = 0$

$\pi + \pi C = 0$

$\pi = -\pi C$

$C = -1$

$y(x) = -x\cos x - x$

(25)

(a) $1/yt = (1/y)(1/t)$ separable

(b) $\sin(y)$ separable

(c) $t\ln(y) + t = t(\ln y + 1)$ separable

(d) $te^{y} + t^{2}$ not separable

(26) $y' = \dfrac{1+3t^2}{3y^2-6y}$ $\quad y(0)=1$

$\dfrac{dy}{dt} = \dfrac{1+3t^2}{3y^2-6y} \Rightarrow (3y^2-6y)\,dy = (1+3t^2)\,dt$

$\Rightarrow \int(3y^2-6y)\,dy = \int(1+3t^2)\,dt$

$\Rightarrow \dfrac{3y^3}{3} - \dfrac{6y^2}{2} = t + \dfrac{3t^3}{3} + C$

$\Rightarrow y^3 - 3y^2 = t + t^3 + C$

$1 - 3 = C$

$-2 = C$

So we have $y^3 - 3y^2 = t + t^3 - 2$

(27)

(a) $y' = \dfrac{3t^2}{2y} \Longleftrightarrow \dfrac{dy}{dt} = \dfrac{3t^2}{2y}$

$\Rightarrow 2y\,dy = 3t^2\,dt$

$\Rightarrow 2\int y\,dy = 3\int t^2\,dt$

$\Rightarrow \dfrac{\cancel{2}y^2}{\cancel{2}} = \dfrac{3t^3}{3} + C = t^3 + C$

$y = \pm\sqrt{t^3 + C}$

(b) $y' = -y^2 \sin t$ $\quad \frac{dy}{dt} = -y^2 \sin t$

$\Rightarrow \int \frac{dy}{-y^2} = \int \sin(t)\,dt$

$\Rightarrow \frac{1}{y} = -\cos(t) + c$

So $y(t) = \frac{1}{c - \cos t}$

(c) $\frac{dy}{dt} = \frac{1+y^2}{t}$

$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dt}{t}$

$\Rightarrow \tan^{-1}(y) = \ln(t) + c$

So $y(t) = \tan(\ln t + c)$

(d) $dy/dt = (y^2 - y)\sin t$

$\Rightarrow \int \frac{dy}{y^2 - y} = \int \sin t\,dt$

$\ln(1-y) - \ln(y) = -\cos(t) + c$

$\Rightarrow e^{\ln(1-y)} e^{-\ln(y)} = e^{-\cos(t) + c}$